STABILITY RESULTS

1. Linear Time Invariant Systems, $\dot{x} = Ax$
   (a) Compute eigenvalues of A. If eigenvalues are in the left half of the real/complex plane, you get (AS) on some neighborhood.
   (b) Solve the Lyap. Matrix Equation for P: $A^T P + PA = -Q$, where P, Q are positive definite matrices. Your Lyap function is then $V = x^T Px$. Generally choose $Q = I$. If P is positive definite, then (GAS). If P is not pd, most likely unstable.
2. Linear Time Varying Systems, $\dot{x} = A(t)x$
   (a) A sufficient, but not necessary condition is if eigenvalues of $A(t)^T + A(t)$ are in the left half of the plane for all $t \Rightarrow$ (ES/AS). However, an (AS) linear time varying system does not imply negative eigenvalues for all time. Then $V = x^T x$. (p. 115)
3. Autonomous Nonlinear Systems, $\dot{x} = f(x)$
   (a) Linearize ($\dot{z} = \frac{\partial f}{\partial x}(x^*)z$) and try Lyap. Indirect Method:
      (i) If the linearized system is (AS) $\Rightarrow$ the nonlinear system is (AS).
      (ii) If the linearized system is unstable $\Rightarrow$ the nonlinear system is unstable.
      (iii) If the linearized system is marginally stable $\Rightarrow$ no result.
   (b) If you have a preponderance of $x_1^2 + x_2^2$, try converting to polar and working with $\dot{r}$. (p. 35)
   (c) If of the form $\ddot{x} + B\dot{x} + f(x) = 0$, write a function $V(x)$ that is about the mechanical energy. Namely, $V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x f(s)ds$.
   (d) If given a Lyap. Function check the following (Lyap. Direct Method)
      (i) Have a (locally) pd function so that $V(0) = 0$ and $V(x) > 0$.
      (ii) If $\dot{V}(x) \leq 0 \Rightarrow$ (S).
      (iii) If $\dot{V}(x) < 0 \Rightarrow$ (AS).
      (iv) If $V(x) \to \infty$ as $\|x\| \to \infty$ *and* the former conditions hold $\forall x \in \mathbb{R}^n$, then (GAS). The radial unbounded conditions are on p. 65.
   (e) How to find a Lyap Function (!)
      (i) Krasovskii's first result: Compute $A(x) = \frac{\partial f}{\partial x}$, the Jacobian. If $F(x) = A(x) + A^T(x)$ is negative definite in some neighborhood of 0, then (AS) and $V(x) = f^T f$. If true $\forall x \in \mathbb{R}^n \Rightarrow$ (GAS). We have a general Krasovskii result that says you can dream up some pd $P(x), Q(x)$ such that $F(x) = A^T P + PA + Q$ is negative definite $\Rightarrow$ (G/AS). Then have $V(x) = f^t P f$.Good luck with that.
      (ii) Variable Gradient Method
         (A) Assume $\nabla V_i = \sum_{j=1}^n a_{ij} x_j$. The curl condition has to hold: $\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$. In $\mathbb{R}^2$, have $a_{12} = a_{21}$. Then choose your a_{ij}'s so that $\dot{V} = \nabla V \cdot \dot{x}$ is negative definite, and $V = \int_0^{x_1} \nabla V_1(x_1, 0)\, dx_1 + \int_0^{x_2} \nabla V_2(x_1, x_2)\, dx_2$ is positive definite. Perhaps try $a_{12} = a_{21} = 0$.
4. Non-Autonomous, Nonlinear Systems, $\dot{x} = f(t, x)$
   (a) Find a Lyap function $V(t, x)$ such that:

(i) V is locally pos def, i.e there exists some pdf $V_0(x)$ s.t $0 < V_0(x) \leq V(t,x)$.

(ii) $\dot{V}$ is negative semi-definite $\Rightarrow$ (S).

(iii) If V is also decrescent, i.e. there exists some $V_1(x)$ so that $V(t,x) < V_1(x)$ then $\Rightarrow$ (US).

(iv) If $\dot{V}$ is negative definite (and V still decrescent), then $\Rightarrow$ (UAS).

(v) If $V(t,x)$ is radially unbounded, and all of the above, then $\Rightarrow$ (GUAS).

5. Instability Results

   (a) You obtain instability on at the zero equilibrium when: In a certain neighborhood of 0, if $\exists$ a descrescent function $V(t,x)$ such that

      (i) $V(t,0) = 0 \; \forall t \geq t_0$.

      (ii) $V(t_0,x)$ is positive for x arbitrarily close to 0, i.e x can take on strictly positive values arbitrarily close to the origin.

      (iii) $\dot{V}(t,x) > 0$