HEAT TRANSFER AND PARAMETER IDENTIFICATION

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In this project we consider the one dimensional propagation of heat through a metal rod. We iteratively develop a mathematical model to describe the dynamics and then construct an inverse problem to identify several physical parameters. Data are collected from both a copper and aluminum rod that have several theromocouples placed axially down the rod to record the temperature at steady state, e.g $t \sim \infty$.

1. Model Derivation

To develop the model we consider two cases: a perfectly insulated metal rod and a non-insulated metal rod.

1.1. Perfectly Insulated Rod.

To formulate the model, we begin by making the following assumptions about the experiment.

(1) Heat is transferred along the x axis only,
(2) The temperature is uniform across the lateral distance,
(3) The rod is perfectly insulated.

We let $u(t,x)$ denote the temperature in degrees Celsius of the rod at time t and spatial location x. The one dimensional heat equation is given by

$$\rho c_p \frac{\partial u(t,x)}{\partial t} = k \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad x \in [0,L], \tag{1}$$

where c_p is the specific heat, ρ is the mass density, k is the thermal conductivity and the rod has length L. Clearly, these parameters are dependent on the material of the rod.

While two of the three assumptions are reasonable given the experiment, the third is not. In fact, the rod used in the experiment is perfectly *uninsulated.* Therefore, it is necessary to consider the axial heat loss/cooling of the rod with Newton's law of cooling.

1.2. Uninsulated Rod.

We work to build a model that accounts for heat loss along all dimensions of the rod; we retain the first two assumptions. The rate of heat accumulation in an infinitesimally small volume ΔV is given by

$$\begin{aligned}\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t} c_p m u \\ &= c_p ab\Delta x \frac{\partial u}{\partial t},\end{aligned}$$

where m is the mass, u is the temperature of the rod, and a, b are the lateral dimensions of the rod. There is another way to compute the rate of heat accumulation in a volume element ΔV. We simply add in the heat loss contributions due to the height and width of the rod.

$$\begin{aligned}\frac{\partial H}{\partial t} &= \big[\text{heat in, } x\big] - \big[\text{heat out, } x+\Delta x\big] - 2\big[\text{heat in, top}\big] - 2\big[\text{heat out, sides}\big] \\ &= kab\frac{\partial u}{\partial x}\bigg|_{t,x} - kab\frac{\partial u}{\partial x}\bigg|_{t,x+\Delta x} - 2bh\Delta x\big(u(t,x)-u_{\text{amb}}\big) - 2ah\Delta x\big(u(t,x)-u_{\text{amb}}\big) \\ &= kab\left[\frac{\partial u}{\partial x}\bigg|_{t,x} - \frac{\partial u}{\partial x}\bigg|_{t,x+\Delta x}\right] - 2(a+b)\Delta x h\big(u(t,x)-u_{\text{amb}}\big),\end{aligned}$$

where u_{amb} is the room temperature and h is a Newton's Law of Cooling constant. We then equate this equation with the expression for $\frac{\partial H}{\partial t}$ and divide out $ab\Delta x$.

$$c_p\rho\frac{\partial u}{\partial t} = k\left[\frac{\frac{\partial u}{\partial x}\big|_{t,x} - \frac{\partial u}{\partial x}\big|_{t,x+\Delta x}}{\Delta x}\right] - 2\frac{a+b}{ab}h\big(u(t,x)-u_{\text{amb}}\big).$$

Taking the limit of $\Delta x \to 0$, we arrive with the following equation which describes the dynamics while accounting for the axial heat loss of an insulated rod.

$$c_p\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u(t,x)}{\partial x^2} - 2\frac{a+b}{ab}h\big(u(t,x)-u_{\text{amb}}\big), \qquad x\in[0,L]. \tag{2}$$

2. Model Solutions

To solve (1) and (2), we consider only the steady state behavior of the equations. Doing so converts the equations to second order ordinary differential equations, as $\frac{\partial u}{\partial t} = 0$. These solutions can be computed in closed form.

When (1) is taken to steady state, we have

$$u''(x) = 0.$$

This equation admits a general linear solution of

$$u(x) = c_1x + c_2, \tag{3}$$

where the constants of integration c_1 and c_2 must be solved for by using the two boundary conditions.

When axial cooling is taken into account in the case of the non-insulated rod, the steady state behavior of (2) is given by

$$ku''(x) - Ah\big(u(x)-u_{\text{amb}}\big) = 0,$$

where $A = \frac{2(a+b)}{ab}$. This equation admits a general solution of

$$u(x) = c_1e^{\alpha x} + c_2e^{-\alpha x} + u_{\text{amb}}, \tag{4}$$

where $\alpha = \sqrt{\frac{Ah}{k}}$ and c_1 and c_2 are again constants of integration that must be solved for using the boundary conditions.

The choice of boundary conditions is of course non-trivial, and should represent the physical conditions of the rod at the respective endpoints. We consider two sets of boundary conditions for each model and then compare the model to the collected data.

2.1. **The $\infty -$ Rod Boundary Condition.** The first set of boundary conditions are given as

$$\begin{aligned} u'(0) &= Q/k, \\ u(L) &= u_{\text{amb}}. \end{aligned} \tag{5}$$

These boundary conditions have an implicit assumption that the rod is of near infinite length and that the heat will not propagate to $x = L$. Applying (5) to (3) gives the closed form solution

$$u(x) = \frac{Q}{k}x + u_{\text{amb}} - \frac{Q}{k}L. \tag{6}$$

Applying (5) to (4) yields the closed form solution

$$u(x) = \left[\frac{Q/k}{\alpha} - \frac{\frac{Q/k}{\alpha}e^{\alpha L}}{2\cosh(\alpha L)}\right]e^{\alpha x} - \left[\frac{\frac{Q/k}{\alpha}e^{\alpha L}}{2\cosh(\alpha L)}\right]e^{-\alpha x} + u_{\text{amb}}. \tag{7}$$

See the Appendix for a complete derivation of (7).

2.2. **Convective Cooling Boundary Condition.** As it turns out, the assumption of an infinitely long rod may not be a realistic assumption. The end of the rod is notably warm during the experiment, requiring us to reformulate the end boundary condition allowing for convective cooling at the end. Instead of a Dirichlet condition at $x = L$, we consider the following Robin condition at $x = L$.

$$\begin{aligned} u'(0) &= Q/k, \\ u'(L) &= \frac{h}{k}\big(u_{\text{amb}} - u(L)\big). \end{aligned} \tag{8}$$

Applying (8) to (3) gives us

$$u(x) = \frac{Q}{k}x - \frac{Q}{k}L - \frac{Q}{h} + u_{\text{amb}}. \tag{9}$$

Lastly, we apply (8) to (4), giving us an equation taking into account axial heat loss across the rod as well as convective cooling at the end. The solution is

$$u(x) = \left[\frac{Q/k}{\alpha} - \frac{\frac{Q/k}{\alpha}e^{\alpha L}(h/k+\alpha)}{e^{\alpha L}(h/k+\alpha) + e^{-\alpha L}(h/k-\alpha)}\right]e^{\alpha x} - \left[\frac{\frac{Q/k}{\alpha}e^{\alpha L}(h/k+\alpha)}{e^{\alpha L}(h/k+\alpha) + e^{-\alpha L}(h/k-\alpha)}\right]e^{-\alpha x} + u_{\text{amb}}. \tag{10}$$

See the Appendix for a full derivation of (10).

3. NUMERICAL SOLUTIONS AND PARAMETER IDENTIFICATION

At the heart of this project is an inverse problem to identify the parameters Q, k and h. If we let $q = [Q\ k\ h]^T$, then we define our goal function as

$$J(q) = r^T r,$$

where $r = \{z_j - u(x_j; q)\}$, a column vector of residuals between the model and collected data z_j at spatial locations x_j. In our experiment, $j = 1 \ldots 15$ and the spatial distance between each thermocouple was measured with calipers. Both the copper and aluminum rod are measured independently. In our code, $J(q)$ is minimized using Nelder-Mead.

In the experiment we attempt to identify q for both a copper rod and an aluminum rod. Note that in (6) there is no assumption of either axial cooling nor cooling at the end point, and so h is not estimated.

In the following discussion, model 1 refers to (6) (no convective BC or axial cooling), model 2 to (9) (no axial cooling, convective BC), model 3 to (7) (axial cooling, no convective BC), and lastly model 4 to (10) (both convective BC and axial cooling).

3.1. **Copper Rod.** As minimizing $J(q)$ is sensitive to the initial iterate, both care and trial and error must be exercised to locate a good starting point. The following initial iterates are used for each of the respective models. q_0^i refers to the initial iterate for model i.

$$q_0^1 = \begin{bmatrix} 10 \\ 493 \end{bmatrix}$$

$$q_0^2 = \begin{bmatrix} -9.48 \\ 1 \\ 2 \end{bmatrix}$$

$$q_0^3 = \begin{bmatrix} -2.8 \times 10^5 \\ 760 \\ 24000 \end{bmatrix}$$

$$q_0^4 = \begin{bmatrix} -1.1 \times 10^8 \\ 3 \times 10^5 \\ 9 \times 10^9 \end{bmatrix}$$

For each model, a $q^\star$ is found to minimize J. The results of the optimization for all four models, along with the data are plotted in Figure 1. The estimated values for Q, k and h in the copper rod for each model are given in Table 3.1.

	Q	k	h	$\frac{Q}{k}$	$J(q^\star)$
Model 1	-19.047	0.18537	n.e.	-104.69	474.25
Model 2	-23.553	0.24982	0.46139	-94.28	432.95
Model 3	-26.602	0.12773	1463.4	-208.27	365.8051
Model 4	-2.8269×10^8	7.6805×10^5	2.4364×10^{10}	-368.06	0.7800

Table 1. Estimated parameter values across all models for the copper rod.

3.2. **Aluminum Rod.** Here the methodology used on the copper rod is repeated for an aluminum rod. Since this optimization is performed after the copper rod optimization, the initial iterates for the aluminum optimization are the $q^\star$ from the respective models in the copper rod case.

The results of the optimization with all four models, along with the collected data are plotted in Figure 2. The estimated values for Q, k and h for each model are given in Table 3.2.

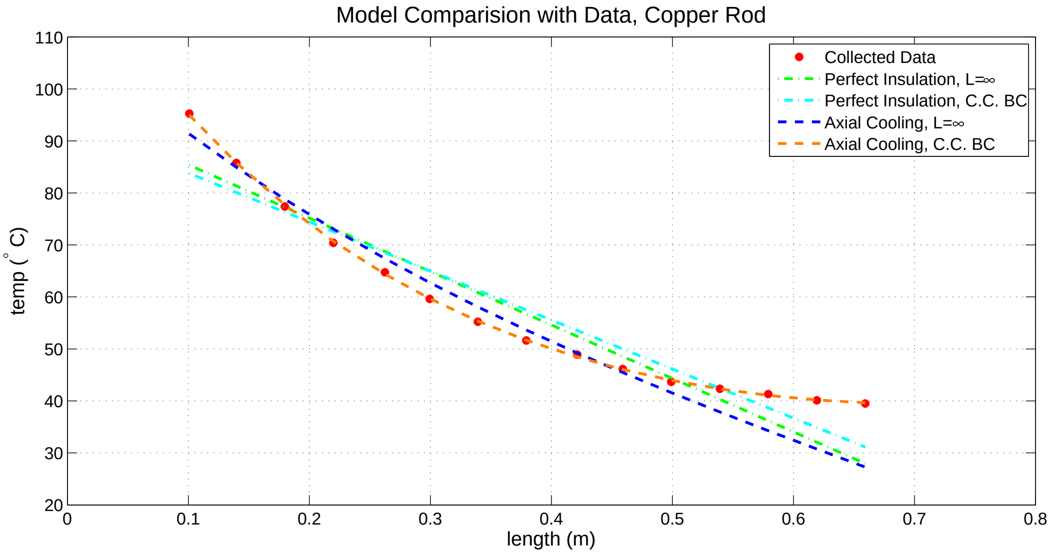

Figure 1. Optimization results with the copper rod.

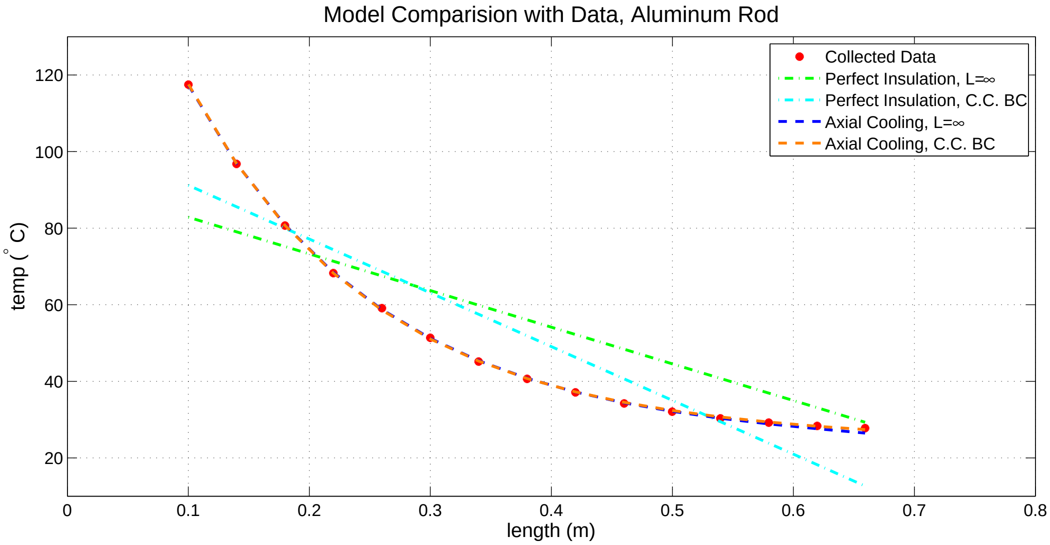

Figure 2. Optimization results with the aluminum rod.

4. Discussion and Analysis

There are several remarks to be made:

(1) The parameters Q, k and h are not uniquely identifiable. As both the solutions and the boundary conditions contain ratios of Q and k, one can only optimize over the ratio of the parameters. If a different initial iterate is chosen, the optimal Q and k values are different.

(2) The linear models are clearly insufficient to describe the data. Given the assumption of perfect insulation is wholly violated by the experiment, this is not surprising. In no case

	Q	k	h	$\frac{Q}{k}$	$J(q^\star)$
Model 1	-18.482	0.19305		-95.737	3011.3
Model 2	-21.312	0.15178	0.6585	-140.41	1858.8
Model 3	-26.602	0.024048	2307.7	-1106.2	2.6722
Model 4	-7.2605×10^6	6519.1	6.3626×10^8	-1113.7	0.89491

Table 2. Estimated parameter values across all models for the aluminum rod.

was a linear fit close to the actual observed dynamics. It is absolutely necessary to account for axial cooling along the metal rod.

(3) The estimated values of thermal conductivity do not match the book values for the copper nor aluminum. This can be due to the non-uniqueness of the estimates, or because the temperature data were not converted to Kelvin, which is the unit that the book values of thermal conductivity are listed.

(4) As expected, the thermal conductivity estimate for aluminum is much lower than that of the copper rod. This is evident and verifiable in the graphs as the temperature range on the aluminum rod is greater than that of copper. A higher thermal conductivity simply means heat is conducted more easily and efficiently than a lower thermal conductivity.

The end point of the aluminum rod is much closer to room temperature, also explaining why model 3, the model with out the convective cooling boundary condition, fits the data very well. Additionally, copper has the second highest thermal conductivity behind only gold. Had we seen a larger k value for aluminum than for copper something would certainly be amiss.

This project emphasizes the iterative nature of the modeling process. Assumptions are built up to facilitate model development and then checked against the reality of the experiment. We note that while some assumptions are benign so as to not affect the quality of the model, others are not and must be adjusted and reformulated.

The final set of model assumptions and boundary conditions, axial cooling along the heated rod and convective cooling at the endpoint, well model the collected data in both the copper and aluminum rods. We conclude that this model can be used to model the steady state temperature distribution in metal rods.

Appendix A. Application of BCs

The solution to the heterogeneous differential equation

$$ku''(x) - Ah\big(u(x) - u_{\text{amb}}\big) = 0$$

can be solved for through the method of undetermined coefficients and is given by

$$u(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} + u_{\text{amb}},$$

where $\alpha = \sqrt{\frac{Ah}{k}}$ and $A = \frac{2(a+b)}{ab}$. The convective cooling boundary conditions are given by

$$\begin{aligned}
\left.\frac{du}{dx}\right|_{x=0} &= \frac{Q}{k} \\
\left.\frac{du}{dx}\right|_{x=L} &= \frac{h}{k}\big(u_{\text{amb}} - u(L)\big).
\end{aligned}$$

The derivative is given by

$$u'(x) = c_1\alpha e^{\alpha x} - c_2\alpha e^{-\alpha x}.$$

Applying the first boundary condition we have

$$\begin{aligned}
u'(0) = c_1\alpha - c_2\alpha &= \frac{Q}{k} \\
= \alpha(c_1 - c_2) &= \frac{Q}{k} \\
\Rightarrow c_1 &= \frac{Q/k}{\alpha} + c_2.
\end{aligned}$$

And for the second BC...

$$\begin{aligned}
u'(L) = c_1\alpha e^{\alpha L} - c_2\alpha e^{-\alpha L} &= \frac{h}{k}\big(u_{\text{amb}} - (c_1 e^{\alpha L} + c_2 e^{-\alpha L} + u_{\text{amb}})\big) \\
c_1\alpha e^{\alpha L} - c_2\alpha e^{-\alpha L} &= \frac{h}{k}\big(-c_1 e^{\alpha L} - c_2 e^{-\alpha L}\big) \\
c_1\alpha e^{\alpha L} - c_2\alpha e^{-\alpha L} + \frac{h}{k}c_1 e^{\alpha L} + \frac{h}{k}c_2 e^{-\alpha L} &= 0 \\
c_1 e^{\alpha L}(h/k + \alpha) + c_2 e^{-\alpha L}(h/k - \alpha) &= 0 \\
\left[\frac{Q/k}{\alpha} + c_2\right] e^{\alpha L}(h/k + \alpha) + c_2 e^{-\alpha L}(h/k - \alpha) &= 0 \\
\frac{Q/k}{\alpha} e^{\alpha L}(h/k + \alpha) + c_2 e^{\alpha L}(h/k + \alpha) + c_2 e^{-\alpha L}(h/k - \alpha) &= 0 \\
c_2\big[e^{\alpha L}(h/k + \alpha) + e^{-\alpha L}(h/k - \alpha)\big] &= -\frac{Q/k}{\alpha} e^{\alpha L}(h/k + \alpha)
\end{aligned}$$

And so

$$\begin{aligned}
\therefore c_2 &= -\frac{\frac{Q/k}{\alpha} e^{\alpha L}(h/k + \alpha)}{e^{\alpha L}(h/k + \alpha) + e^{-\alpha L}(h/k - \alpha)} \\
\therefore c_1 &= \frac{Q/k}{\alpha} + c_2.
\end{aligned}$$

Now let us consider the infinite rod length case.

$$\begin{aligned}
\left.\frac{du}{dx}\right|_{x=0} &= \frac{Q}{k} \\
\left.\frac{du}{dx}\right|_{x=L} &= u_{\text{amb}}.
\end{aligned}$$

From the previous work we have

$$c_1 = \frac{Q/k}{\alpha} + c_2.$$

Working with the end boundary condition, we then have

$$\begin{aligned}
u(L) &= u_{\text{amb}} \\
c_1 e^{\alpha L} + c_2 e^{-\alpha L} + u_{\text{amb}} &= u_{\text{amb}} \\
c_1 e^{\alpha L} + c_2 e^{-\alpha L} &= 0 \\
\left[\frac{Q/k}{\alpha} + c_2\right] e^{\alpha L} + c_2 e^{-\alpha L} &= 0 \\
\frac{Q/k}{\alpha} e^{\alpha L} + c_2 e^{\alpha L} + c_2 e^{-\alpha L} &= 0 \\
\frac{Q/k}{\alpha} e^{\alpha L} + 2c_2 \cosh(\alpha L) &= 0
\end{aligned}$$

And so

$$\begin{aligned}
\therefore c_2 &= -\frac{\frac{Q/k}{\alpha} e^{\alpha L}}{2\cosh(\alpha L)} \\
\therefore c_1 &= \frac{Q/k}{\alpha} + c_2.
\end{aligned}$$